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THE REIDEMEISTER SPECTRUM OF ZM-GROUPS

MARIUS TĂRNĂUCEANU

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Abstract

Given a group G and an automorphism φ of G, two elements $x, y \in G$ are said to be φ-conjugate if $x = gy\varphi(g)^{-1}$ for some $g \in G$. The number $R(\varphi)$ of equivalence classes with respect to this relation is called the Reidemeister number of φ and the set $\{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\}$ is called the Reidemeister spectrum of G. We determine the Reidemeister spectrum of ZM-groups, extending some results of Senden ['The Reidemeister spectrum of split metacyclic groups', Preprint, 2022, arXiv:2109.12892].

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1. Introduction

Let G be a group and $\mathrm{Aut}(G)$ be the group of automorphisms of G. The concept of conjugacy in G can be naturally generalised to the so-called *twisted conjugacy*: given $\varphi \in \mathrm{Aut}(G)$, two elements $x, y \in G$ are called *φ-conjugate* if there exists $g \in G$ such that $x = gy\varphi(g)^{-1}$. We define the *Reidemeister number* $R(\varphi)$ of φ as the number of equivalence classes with respect to this relation. Note that $R(\varphi) \in \mathbb{N}_0 \cup \{\infty\}$. Also, we define the *Reidemeister spectrum* of G to be $\mathrm{Spec}_R(G) = \{R(\varphi) \mid \varphi \in \mathrm{Aut}(G)\}$. If $\mathrm{Spec}_R(G) = \{\infty\}$, we say that G has the R_∞-property.

Twisted conjugacy arises naturally in Nielsen fixed-point theory (see, for example, [12]), but there is also a strong algebraic interest in its study, in particular, in the study of the R_∞-property (see, for example, [5, 8, 9, 15]). For many groups, either partial or full information is known about their Reidemeister spectrum. It has either been proven that they have the R_∞-property (such as Baumslag–Solitar groups [7] or Thompson's group [1]) or that they do not (such as certain groups of exponential growth [10] or free groups of infinite rank [3]), and for some groups, even the complete Reidemeister spectrum has been determined (such as low-dimensional crystallographic groups [4] or finite abelian groups [14]).

The starting point for our discussion is the paper [13], where the Reidemeister spectrum has been determined for split metacyclic groups of the form $(C_n \times C_{p^m}) \rtimes C_p$ where p is prime, m and n are nonnegative integers, and n is coprime with p. In what follows, we will deal with ZM-groups, that is, finite groups with all Sylow subgroups cyclic. By [11, Ch. IV, Satz 2.11], such a group is of type

$$\mathrm{ZM}(m,n,r) = \langle a,b \mid a^m = b^n = 1,\ b^{-1}ab = a^r\rangle,$$

where the triple (m,n,r) satisfies the conditions

$$(m,n) = (m,r-1) = 1 \quad \text{and} \quad r^n \equiv 1 \pmod m.$$

Throughout the paper, for a finite number of positive integers $a_1, a_2, \ldots, a_n$, we denote by $(a_1, a_2, \ldots, a_n)$ their greatest common divisor. It is clear that $|\mathrm{ZM}(m,n,r)| = mn$ and $Z(\mathrm{ZM}(m,n,r)) = \langle b^{o_m(r)}\rangle$, where

$$o_m(r) = \min\{k \in \mathbb{N}^* \mid r^k \equiv 1 \pmod m\}$$

is the multiplicative order of r modulo m. For simplicity of notation, throughout the paper, we will use d instead of $o_m(r)$. For every nonnegative integer u, we define

$$[u]_r = \begin{cases} 1 + r + \cdots + r^{u-1}, & u > 0,\\ 0, & u = 0.\end{cases}$$

Our main result is stated as follows.

Theorem 1.1. *We have*

$$\mathrm{Spec}_R(\mathrm{ZM}(m,n,r)) = \left\{\frac{1}{m}\sum_{v=0}^{m-1} \sum_{\substack{0\le u\le n-1\\ n/(n,y-1)\mid u}} \frac{(m,[u]_r)}{o_{(m,[u]_r)/(m,[u]_r,v)}(r)} \;\middle|\; 0 \le y < n, y \equiv 1 \pmod d\right\}.$$

Example 1.2. For the dicyclic group $\mathrm{Dic}_3 = \mathrm{ZM}(3,4,2) = \mathrm{SmallGroup}(12,1)$, we have $d = 2$ and the above formula leads to $\mathrm{Spec}_R(\mathrm{Dic}_3) = \{4,6\}$.

The following is a particular case of Theorem 1.1.

Corollary 1.3. *If n is prime, then*

$$\mathrm{Spec}_R(\mathrm{ZM}(m,n,r)) = \left\{n - 1 + \frac{S}{n}\right\},$$

where

$$S = \sum_{u=0}^{n-1} (m,[u]_r).$$

Example 1.4. For $n = 2$ and $r = m-1$, we have $\mathrm{ZM}(m,n,r) \cong D_{2m}$, the dihedral group of order $2m$. Then, $S = m+1$ and so the formula in Corollary 1.3 becomes

$$\mathrm{Spec}_R(D_{2m}) = \left\{\frac{m+3}{2}\right\} \quad \text{for all odd integers } m \ge 3.$$

We remark that a split metacyclic group $(C_n \times C_{p^m}) \rtimes C_p$ becomes a ZM-group for $m = 0$ and so Theorem 1.1 extends the results obtained in [13] in this particular case.

Our approach uses some basic tools of group action theory, such as Burnside's lemma, and modular arithmetic.

BURNSIDE'S LEMMA. *Let G be a finite group acting on a finite set X and let*

$$\mathrm{Fix}(g) = \{x \in X \mid g \circ x = x\} \quad \text{for all } g \in G.$$

Then, the number of distinct orbits is

$$k = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|.$$

Most of our notation is standard and will not be repeated here. Basic definitions and results on group theory can be found in [11].

2. Proofs of the main results

The general form of automorphisms of a metacyclic group has been determined in the main result of [2]. For ZM-groups, we obtained the following result in [6].

THEOREM 2.1. *Each automorphism of* $\mathrm{ZM}(m, n, r)$ *is given by*

$$b^u a^v \mapsto b^{yu} a^{x_1 v + x_2 [u]_r}, \quad u, v \geq 0,$$

for a unique triple of integers (x_1, x_2, y) *such that*

$$0 \leq x_1, x_2 < m, \quad (x_1, m) = 1, \quad 0 \leq y < n \quad \text{and} \quad y \equiv 1 (\mathrm{mod}\ d). \tag{2.1}$$

In particular,

$$|\mathrm{Aut}(\mathrm{ZM}(m, n, r))| = m\varphi(m)\frac{n}{d}.$$

In what follows, we will denote by $f_{(x_1, x_2, y)}$ the automorphism of $\mathrm{ZM}(m, n, r)$ determined by a triple (x_1, x_2, y) satisfying (2.1).

We are now able to prove our results.

PROOF OF THEOREM 1.1. Let $f = f_{(x_1, x_2, y)} \in \mathrm{Aut}(\mathrm{ZM}(m, n, r))$. To compute $R(f)$, we will apply Burnside's lemma to the action of $\mathrm{ZM}(m, n, r)$ on $\mathrm{ZM}(m, n, r)$ given by

$$g \circ x = gxf(g)^{-1} \quad \text{for all } g, x \in \mathrm{ZM}(m, n, r).$$

Let $g = b^u a^v$ and $x = b^\alpha a^\beta$, where $0 \leq u, \alpha < n$ and $0 \leq v, \beta < m$. From Theorem 2.1,

$$\begin{aligned}
\mathrm{Fix}(g) &= \{x \in \mathrm{ZM}(m, n, r) \mid f(g) = x^{-1} g x\} \\
&= \{b^\alpha a^\beta \in \mathrm{ZM}(m, n, r) \mid b^{yu} a^{x_1 v + x_2 [u]_r} = b^u a^{r^\alpha v - \beta(r^u - 1)}\} \\
&= \{b^\alpha a^\beta \in \mathrm{ZM}(m, n, r) \mid b^{yu} = b^u \text{ and } a^{x_1 v + x_2 [u]_r} = a^{r^\alpha v - \beta(r^u - 1)}\}.
\end{aligned}$$

We have

$$b^{yu} = b^u \quad \Leftrightarrow \quad n \mid (y-1)u \quad \Leftrightarrow \quad \frac{n}{(n, y-1)} \,\Big|\, u$$

and

$$a^{x_1 v + x_2 [u]_r} = a^{r^\alpha v - \beta(r^u - 1)} \quad \Leftrightarrow \quad m \mid ((r^\alpha - x_1)v - \beta(r^u - 1) - x_2[u]_r).$$

For fixed u, v and α, the last congruence has solutions β if and only if

$$(m, [u]_r) \mid (r^\alpha - x_1)v. \tag{2.2}$$

Moreover, if it has solutions, then there are exactly $(m, [u]_r)$ solutions. We also remark that (2.2) is equivalent to

$$\frac{(m, [u]_r)}{(m, [u]_r, v)} \,\Big|\, r^\alpha - x_1 \tag{2.3}$$

and (2.3) has the same number of solutions α as

$$\frac{(m, [u]_r)}{(m, [u]_r, v)} \,\Big|\, r^\alpha - 1,$$

that is, as

$$o_{(m,[u]_r)/(m,[u]_r,v)}(r) \mid \alpha. \tag{2.4}$$

To simplify the notation, we write $w = o_{(m,[u]_r)/(m,[u]_r,v)}(r)$. Obviously, (2.4) has n/w solutions and so

$$\begin{aligned}
R(f) &= \frac{1}{mn} \sum_{g \in \mathrm{ZM}(m,n,r)} |\mathrm{Fix}(g)| \\
&= \frac{1}{mn} \sum_{(u,v)} |\{(\alpha, \beta) \mid b^{yu} = b^u \text{ and } a^{x_1 v + x_2[u]_r} = a^{r^\alpha v - \beta(r^u - 1)}\}| \\
&= \frac{1}{mn} \sum_{v=0}^{m-1} \sum_{\substack{0 \le u \le n-1 \\ n/(n,y-1) \mid u}} \sum_{\alpha=0}^{n-1} |\{\beta = \overline{0, m-1} \mid m \mid ((r^\alpha - x_1)v - \beta(r^u - 1) - x_2[u]_r)\}| \\
&= \frac{1}{mn} \sum_{v=0}^{m-1} \sum_{\substack{0 \le u \le n-1 \\ n/(n,y-1) \mid u}} \sum_{\substack{0 \le \alpha \le n-1 \\ (m,[u]_r) \mid (r^\alpha - x_1)v}} (m, [u]_r) \\
&= \frac{1}{mn} \sum_{v=0}^{m-1} \sum_{\substack{0 \le u \le n-1 \\ n/(n,y-1) \mid u}} (m, [u]_r)\, |\{\alpha = \overline{0, n-1} \mid (m, [u]_r) \mid (r^\alpha - x_1)v\}| \\
&= \frac{1}{mn} \sum_{v=0}^{m-1} \sum_{\substack{0 \le u \le n-1 \\ n/(n,y-1) \mid u}} (m, [u]_r)\, \frac{n}{w} \\
&= \frac{1}{m} \sum_{v=0}^{m-1} \sum_{\substack{0 \le u \le n-1 \\ n/(n,y-1) \mid u}} \frac{(m, [u]_r)}{w},
\end{aligned}$$

as desired. □

PROOF OF COROLLARY 1.2. If n is prime, then we have $d = n$, and so the conditions $0 \leq y < n$ and $y \equiv 1 \pmod d$ imply $y = 1$ for any $f = f_{(x_1,x_2,y)} \in \mathrm{Aut}(\mathrm{ZM}(m,n,r))$. Thus, again writing $w = o_{(m,[u]_r)/(m,[u]_r,v)}(r)$, we have

$$R(f) = \frac{1}{m}\sum_{v=0}^{m-1}\sum_{u=0}^{n-1}\frac{(m,[u]_r)}{w}.$$

Since w is a divisor of n, either $w = 1$ or $w = n$. We remark that

$$w = 1 \quad\Leftrightarrow\quad \frac{(m,[u]_r)}{(m,[u]_r,v)} = 1 \quad\Leftrightarrow\quad (m,[u]_r) \mid v.$$

It follows that

$$\begin{aligned}
R(f) &= \frac{1}{m}\sum_{v=0}^{m-1}\sum_{u=0}^{n-1}\frac{(m,[u]_r)}{w}\\
&= \frac{1}{m}\sum_{u=0}^{n-1}(m,[u]_r)\Bigg(\sum_{\substack{0\leq v\leq m-1\\(m,[u]_r)\mid v}} 1 + \sum_{\substack{0\leq v\leq m-1\\(m,[u]_r)\nmid v}}\frac{1}{n}\Bigg)\\
&= \frac{1}{m}\sum_{u=0}^{n-1}(m,[u]_r)\bigg[\frac{m}{(m,[u]_r)} + \frac{1}{n}\Big(m - \frac{m}{(m,[u]_r)}\Big)\bigg]\\
&= \sum_{u=0}^{n-1}\Big(1-\frac{1}{n}\Big) + \frac{1}{n}\sum_{u=0}^{n-1}(m,[u]_r)\\
&= n - 1 + \frac{S}{n},
\end{aligned}$$

completing the proof. □

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MARIUS TĂRNĂUCEANU, Faculty of Mathematics,
'Al.I. Cuza' University, Iaşi, Romania
e-mail: tarnauc@uaic.ro